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LETTER TO THE EDITOR

Feynman path-integral representations for the classical harmonic oscillator with stochastic frequency

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Abstract

We propose a Feynman path-integral solution for classical harmonic oscillator motions with stochastic frequency.

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Introduction

The problem of the (random) motion of a harmonic oscillator in the presence of a stochastic time-dependent perturbation on its frequency is of great theoretical and pratical importance [1, 2]. In this Letter we propose a formal path-integral solution for the above-mentioned problem by closely following our previous studies [3, 4]. In section 1 we write a Feynman path-integral representation for the external forcing problem. In section 2 we consider a similar problem for the initial-condition case.

1. The Green function for external forcing

Let us start our analysis by considering the classical motion equation of a harmonic oscillator subject to an external forcing

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + w_0^2(1+g(t))\right\} x(t) = F(t). \tag{1}$$

Here $w_0^2(1+g(t))$ is the time-dependent frequency with stochastic part given by the random function $g(t)$ obeying the Gaussian statistics

$$\langle g(t)g(t')\rangle = K(t,t'). \tag{2}$$

The solution of equation (1) is, thus, given by

$$x(t,[g]) = \int_0^t G(t,t',[g])F(t')\,\mathrm{d}t' \tag{3}$$

where $G(t,t',[g])$ denotes the Green problem functionally depending on the random frequency $g(t)$ and the notation emphasizes that the objects under study are functionals of the random part $g(t)$ of the harmonic oscillator frequency.

In order to write a path-integral representation for the Green function equation (3) we follow our previous study [3] by using a 'proper-time' technique by introducing related Schrödinger wave equations with an initial point source and $-\infty \leqslant t \leqslant +\infty$:

$$\mathrm{i}\frac{\partial \bar{G}(s;(t,t'))}{\partial s} = -\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + w_0^2(1+g(t))\right]\bar{G}(s;(t,t')) \tag{4}$$

$$\lim_{s\to 0}\bar{G}(s;(t,t')) = \delta(t-t') \tag{5}$$

$$\lim_{s\to\infty}\bar{G}(s;(t,t')) = 0. \tag{6}$$

At this point we note the following identity between the Schrödinger equations (4)–(6) and the searched harmonic oscillator Green function:

$$G[(t,t',[g]) = -\mathrm{i}\int_0^\infty \mathrm{d}s\,\bar{G}(s;(t,t')). \tag{7}$$

Let us, thus, write a path integral for the associated Schrödinger equations (4)–(7) by considering $\bar{G}(s;(t,t'))$ in the operator form (the Feynman–Dirac propagator):

$$\bar{G}(s;(t,t')) = \langle t|\exp(\mathrm{i}sH)|t'\rangle \tag{8}$$

where H is the differential operator

$$H = -\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + w_0^2(1+g(t))\right\}. \tag{9}$$

As in quantum mechanics we write equation (8) as an infinite product of short-time s propagations

$$\langle t|\exp(\mathrm{i}sH)|t'\rangle = \lim_{N\to\infty}\prod_{i=1}^{N}\int_{-\infty}^{+\infty}\mathrm{d}t_i\,\langle t_i|\exp\mathrm{i}\left(\frac{s}{N}H\right)|t_{i-1}\rangle. \tag{10}$$

The standard short-time expansion in the s-parameter for equation (10) is given by

$$\lim_{s\to 0^+}\langle t_i|\mathrm{e}^{\mathrm{i}sH}|t_{i-1}\rangle = \lim_{s\to 0^+}\int \mathrm{d}w_i\;\exp\{\mathrm{i}s[w_i^2 + w_0^2(1+g^2(t_{i-1}))]\}\exp[\mathrm{i}w_i(t_i - t_{i-1})]. \tag{11}$$

If we substitute equation (11) into (10) and take the Feynman limit of $N\to\infty$, we will obtain the following path-integral representation after evaluating the w_i-Gaussian integrals of the representation equation (8):

$$\bar{G}(s;(t,t')) = \int\Bigg(\prod_{\substack{0<\sigma<s\\ t(0)=t';t(s)=t}}\mathrm{d}t\,(\sigma)\Bigg)\exp\left\{\frac{\mathrm{i}}{2}\int_0^2\mathrm{d}\sigma\left[\left(\frac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma}\right)^2\right]\right\}$$
$$\times\exp\left\{\mathrm{i}\int_0^s[w_0^2(1+g(t(\sigma)))]\,\mathrm{d}\sigma\right\}. \tag{12}$$

The averaged out equation (7) is thus given straightforwardly by the following Feynman polaron-like path integral:

$$\langle G(t,t',[g])\rangle_g = -\mathrm{i}\int_0^\infty \mathrm{d}s\,(\mathrm{e}^{\mathrm{i}w_0^2 s})\int_{t(0)=t';t(s)=t}D^{\mathrm{F}}[t(\sigma)]\exp\left\{\frac{\mathrm{i}}{2}\int_0^s\mathrm{d}\sigma\left[\left(\frac{\mathrm{d}t}{\mathrm{d}\sigma}\right)^2\right]\right\}$$
$$\times\exp\left\{-w_0^4\int_0^s\mathrm{d}\sigma\int_0^s\mathrm{d}\sigma'\,K(t(\sigma);t(\sigma'))\right\}. \tag{13}$$

The two-point correlation function is still given by a two-full similar path integral, namely

$$\langle G(t_1, t_1', [g])G(t_2, t_2', [g])\rangle_g = \int_0^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{e}^{\mathrm{i}w_0^2(s_1+s_2)}$$
$$\times \int_{t_1(0)=t_1';t_2(s_1)=t_1;t_2(0)=t_2';t_2(s_2)=t_2} D^{\mathrm{F}}[t_1(\sigma), t_2(\sigma)]$$
$$\times \exp\left\{\frac{\mathrm{i}}{2}\left(\int_0^{s_1} \mathrm{d}\sigma\, (\dot{t}_1(\sigma))^2 + \int_0^{s_2} \mathrm{d}\sigma\, (\dot{t}_2(\sigma))^2\right)\right\}$$
$$\times \exp\left\{ - w_0^4\left[\int_0^{s_1} \mathrm{d}\sigma \int_0^{s_1} \mathrm{d}\sigma'\, K(t_1(\sigma), t_1(\sigma')) + \int_0^{s_1} \mathrm{d}\sigma \int_0^{s_2} \mathrm{d}\sigma'\right.\right.$$
$$\left.\left.\times (K(t_1(\sigma), t_2(\sigma')) + K(t_2(\sigma), t_1(\sigma'))) + \int_0^{s_1} \mathrm{d}\sigma \int_0^{s_2} \mathrm{d}\sigma'\, K(t_2(\sigma), t_2(\sigma'))\right]\right\}. \tag{14}$$

Similar N-iterated path-integral expressions hold true for the N-point correlation function $\langle x(t_1, [g]) \cdots x(t_N, [g])\rangle_g$. Explicit and approximate evaluations of the path-integral equations (13) and (14) follow procedures similar to those used in the usual contexts of physics statistics, quantum mechanics and random wave propagation (last reference of [1]).

Let us show such exact integral representation for equation (13) in the case of the practical case of a slowly varying (even function) kernel of the form

$$\begin{aligned} &K(t) \sim K(0) - \frac{\ell_0}{2}\,|t|^2 \qquad |t| \ll \left(+\frac{K(0)}{\ell_0}\right)^{1/2} = L \\ &K(t) \sim 0 \qquad |t| \gg L. \end{aligned} \tag{15}$$

In this case, we have the following exact result for the path integral in equation (13):

$$\langle G(t, t', [g])\rangle_g = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s(-\mathrm{i}w_0^2)} \mathrm{e}^{-(w_0^4 K(0))s^2} \left\{ (2\pi \mathrm{i}s)^{\frac{1}{2}} \left[\frac{w_0^2(-\frac{\ell_0}{2})^{\frac{1}{2}} S^{\frac{3}{2}}}{\mathrm{sen}[s^{\frac{3}{2}} w_0^2(-\frac{\ell_0}{2})^{\frac{1}{2}}]}\right]\right.$$
$$\left.\times \exp\left(\left[\frac{\mathrm{i}w_0^2}{2} s^{\frac{1}{2}} \left(-\frac{\ell_0}{2}\right)^{\frac{1}{2}} \cot\left(w_0^2 \left(-\frac{\ell_0}{2}\right)^{\frac{1}{2}} s^{\frac{3}{2}}\right)\right] (t-t')^2\right)\right\}. \tag{16}$$

Another useful formula is that related to the 'mean-field' averaged path integral when the kernel $K(t, t')$ has a Fourier transform of the general form

$$K(t, t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}p \cdot \mathrm{e}^{\mathrm{i}p(t-t')} \tilde{K}(p). \tag{17}$$

The envisaged integral representation for equation (13) is, thus, given by

$$\langle G(t, t', [g])\rangle_g = -\mathrm{i} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{\mathrm{i}sw_0^2} \exp\left\{ - \frac{w_0^4}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{K}(\rho) \times M(p, s, t, t')\right\} \tag{18}$$

where

$$M(p, s, t, k') = \int_0^s \mathrm{d}\sigma \int_0^s \mathrm{d}\sigma' \left\{ \int_{\substack{t(0)=t' \\ t(s)=t}} D^{\mathrm{F}}[t(\sigma)] \exp\left\{\frac{\mathrm{i}}{2}\int_0^s \left[\left(\frac{\mathrm{d}t}{\mathrm{d}\sigma}\right)\right]^2\right\}\right.$$
$$\left.\times \exp[\mathrm{i}p(t(\sigma) - t(\sigma'))]\right\}. \tag{19}$$

2. The homogeneous problem

Let us start this section by considering now the problem of determining two linearly independent solutions of the homogeneous harmonic oscilator problem

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + w_0^2(1 + g(t))\right\} x(t) = 0 \tag{20}$$

with the initial conditions

$$x(0) = x_0 \qquad x'(0) = v_0. \tag{21}$$

It is straightforward to see that two linearly independent solutions are given by the following expressions:

$$x_1(t, [g]) = \exp\left\{\int_0^t y^2(\sigma, [g])\,\mathrm{d}\sigma\right\} \tag{22}$$

$$x_2(t, [g]) = x_1(t, [g]) \int_0^t (x_1(\sigma, [g]))^{-2}\,\mathrm{d}\sigma \tag{23}$$

where $y(t, [g])$ satisfy the first-order nonlinear ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) + (y(t)^2 = -w_0^2(1 + g(t)). \tag{24}$$

In order to obtain a path-integral representation for equation (24) we remark that the whole averaging (stochastic) information is contained in the characteristic functional

$$Z[j(t)] = \left\langle \exp\left\{\mathrm{i}\int_0^\infty \mathrm{d}t\; y(t, [g])\, j(t)\right\}\right\rangle_g . \tag{25}$$

In order to write a path-integral representation for the characteristic functional equation (25) we rewrite (25) as a Gaussian functional integral in $g(t)$:

$$Z[j(t)] = \int D^{\mathrm{F}}[g(t)] \exp\left(-\tfrac{1}{2}\int_0^\infty \mathrm{d}t\,\mathrm{d}t' g(t) K^{-1}(t, t') g(t')\right) \exp\left\{\mathrm{i}\int_0^\infty \mathrm{d}t\; y(t, [g])\right\}. \tag{26}$$

At this point we observe the validity of the following functional integral representation for the characteristic functional equation (26) after considering the functional change $g(t) \to y(t)$ defined by equation (24), namely

$$\begin{aligned} Z[j(t)] = \int D^{\mathrm{F}}[y(t)] \times \exp\left(-\frac{1}{2(w_0)^4}\int_0^\infty \mathrm{d}t\,\mathrm{d}t'\left[\left(\frac{\mathrm{d}y}{\mathrm{d}t} + y^2\right)(t) + w_0^2\right]\right) K(t, t') \\ \times \left[\left(\frac{\mathrm{d}y}{\mathrm{d}t'} + y^2\right)(t') + w_0^2\right] \exp\left\{\mathrm{i}\int_0^\infty \mathrm{d}t\; j(t) y(t)\right\} \end{aligned} \tag{27}$$

where we have used the fact that the Jacobian associated with the functional change $g(t) \to y(t)$ is unity:

$$\det{}_{\mathrm{F}}\left[\frac{\mathrm{d}}{\mathrm{d}t} + 2y\right] = \frac{\delta g(t)}{\delta y(t)} = 1. \tag{28}$$

At this point it is instructive to remark that in the important case of a white-noise frequency process with strength γ

$$K(t, t') = \gamma\delta(t - t) \tag{29}$$

the path-integral representation for the characteristic functional equation (27) takes the more amenable form

$$Z[j(t)] = \int D^{\mathrm{F}}[y(t)] \exp\left\{ -\frac{\gamma}{2(w_0)^4} \int_0^\infty \mathrm{d}t \left[\frac{\mathrm{d}y}{\mathrm{d}t} + y^2(t) + w_0^2 \right]^2 \right\} \times \exp\left\{ \mathrm{i} \int_0^\infty \mathrm{d}t\, y(t) j(t) \right\}; \tag{30}$$

we obtain, thus, the standard $\lambda\varphi^4$ zero-dimensional path integral as a functional integral representation for the characteristic function equation (25) in the white-noise case

$$Z[j(t)] = \int D^{\mathrm{F}}[\bar{y}(t)] \exp\left\{ -\frac{\gamma}{2(w_0)^4} \int_0^\infty \mathrm{d}t \left[\left(\frac{\mathrm{d}\bar{y}}{\mathrm{d}t} \right)^2 + 2w_0^2 \bar{y}^2 + \bar{y}^4 \right](t) \right\} \times \exp\left\{ \mathrm{i} \int_0^\infty \mathrm{d}t\, \bar{y}(t) j(t) \right\}. \tag{31}$$

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Appendix A

In this appendix we discuss some mathematical points related to the final condition on the quantum propagator equations (4)–(6) (its vanishing in the limit $s \to \infty$). Let us first remark that by imposing the trace class condition on the correlation equation (2) $k(t, t')$ ($\int_0^\infty k(t, t)\,\mathrm{d}t < \infty$) one has the result that all realizations (sampling) $g(t)$ of the associated stochastic process are square integrable functions by a direct application of the Minlos theorem on the domain of functional integrals [5]. At this point, we note that if one restricts $g(t)$ further to be a $\frac{\mathrm{d}^2}{\mathrm{d}t^2}$-perturbation, namely with $w_0^2 \in L^\infty$ and $g(t) \in L^2(0, \infty)$

$$\|(w_0^2(1 + g(t))h\|_{L^2} \leqslant a \left\| \frac{\mathrm{d}^2}{\mathrm{d}t^2} h \right\|_{L^2} + b\|h\|_{L^2} \tag{A.1}$$

and $0 < a < 1$ and b arbitrary, one can apply the Kato–Rellich theorem [6] to be sure that the domain of the differential operator $-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + w_0^2 + w_0^2 g(t)$ is (at least) contained on the domain of $-\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ which in turn has a purely continuous spectrum on $L^2(R)$. As a consequence of the above exposed remarks one does not have bound states on the Schrödinger operator spectrum of equation (4) for each realization of $g(t)$ on the above-cited functional class. As a consequence, we have that the evolution operator

$$\exp\left(\mathrm{i}s \left[-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + w_0^2(1 + g(t)) \right] \right) \tag{A.2}$$

is a unitary operator on $\mathrm{Dom}(-\frac{\mathrm{d}^2}{\mathrm{d}t^2})$ and $\overline{G}(s, (t, t'))$ in turn is expected to have the same behaviour of $\exp(\mathrm{i}s[-\frac{\mathrm{d}^2}{\mathrm{d}t^2}])$ (asymptotic completeness [7]) at $s \to \infty$, which in turn vanishes at $s \to \infty$, making our condition equation (6) highly reasonable from a mathematical physicist's point of view.

At this point we remark that the *different problem* of *determining* the *quantum propagator* of a particle under the influence of a harmonic potential $V(x) = \frac{1}{2}\, w_0^2 x^2$ and a stochastic potential $g(x)$ may be possible to handle in our framework. Note that the 'unperturbed' Hamiltonian $-\frac{\hbar^2 \mathrm{d}^2}{2m\, \mathrm{d}x^2} + \frac{1}{2} m w_0^2 x^2$ has a pure point spectrum (bound states) and its perturbation by a $g(x) \in L^2(R)$ potential does not alter the spectrum behaviour. However, one can proceed in a mathematical physicist's (formal) way as exposed in our Letter and see that equation (3) adapted to this case is still (formally) correct. Namely

$$\langle G(t,(x,x'))\rangle_g = \int \left(\prod_{\substack{0<\sigma<t \\ x(0)=x';x(t)=x}} \mathrm{d}x\,(\sigma) \right) \exp\left\{ \frac{\mathrm{i}}{2} m \int_0^t \mathrm{d}\sigma \left[\left(\frac{\mathrm{d}x(\sigma)}{\mathrm{d}\sigma} \right)^2 \right] \right\}$$

$$\times \exp\left\{ \frac{\mathrm{i}}{2} m w_0^2 \int_0^t \mathrm{d}\sigma [x(\sigma)]^2 \right\} \exp\left\{ -w_0^4 \int_0^t \mathrm{d}\sigma \int_0^t \mathrm{d}\sigma' \, K(x(\sigma); x(\sigma)) \right\}. \tag{A.3}$$

Finally we want to point out that in the general case where $k(t,t')$ is not defined on an operator of the trace class, the realizations $g(t)$ will be distributional objects. Unfortunately in this case there is no rigorous spectral perturbation mathematics for differential operators acting on nuclear spaces $(S'(R), D'(R))$ etc, which is the natural mathematical setting to understand equation (1) of this Letter.

Appendix B

In this appendix we complete our study by considering the problem of a harmonic oscillator in the presence of a damping term $(0 < t < \infty)$

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \nu \frac{\mathrm{d}}{\mathrm{d}t} + w_0^2(1+g(t)) \right\} x(t) = F(t). \tag{B.1}$$

In order to map the above-written differential equation in the analysis presented in section 1 of this Letter, we implement in equation (B.1) the following time-variable change [3]:

$$\begin{aligned} \zeta &= \frac{m}{\nu}(1-\mathrm{e}^{-(\frac{\nu}{m})t}) \\ y(\zeta) &= x(t). \end{aligned} \tag{B.2}$$

We obtain, thus, the following pure harmonic oscillator differential equation without the damping term in place of the original equation (B.1), namely

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + w_0^2[1+\tilde{g}(\zeta)] \right\} y(\zeta) = \tilde{F}(\zeta). \tag{B.3}$$

Here

$$\tilde{g}(\zeta) = g\left(-\left(\frac{m}{\nu}\right) \lg\left(1 - \left(\frac{\nu}{m}\right)\zeta\right)\right) \tag{B.4}$$

$$\tilde{F}(\zeta) = F\frac{(-(\frac{m}{\nu})\lg(1-(\frac{\nu}{m})\zeta))}{(1-(\frac{\nu}{m})\zeta)^2} \tag{B.5}$$

and the correlator stochastic frequency

$$K(\zeta,\zeta') = K((\zeta,\zeta')) = K\left(\frac{m}{\nu} \left| \lg\left[\frac{(1-(\frac{\nu}{m})\zeta)}{(1-(\frac{\nu}{m})\zeta')} \right] \right| \right). \tag{B.6}$$

From here on the analysis goes as in the bulk of this Letter under the condition that the range associated with this new time variable is the finite interval $[0, \frac{m}{\nu}]$.

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